VARIATIONAL RESOLUTION OF WIND COMPONENTS [1]

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ABSTRACT

A variational formulation minimizing the square of velocity differences is used to determine boundary conditions for determination of velocity potential and stream function for the horizontal wind. The boundary velocity is conserved locally, as are the areal total velocity, divergence, and vorticity fields.

1. INTRODUCTION

The horizontal wind $\mathbf{V}$ can be separated into solenoidal and irrotational components by means of the Helmholtz theorem

$$\mathbf{V}=\nabla\chi+\mathbf{k}\times\nabla\psi, \tag{1}$$

where χ is the velocity potential, ψ the stream function, and $\mathbf{k}$ the unit normal to the plane. The velocity potential and stream function are determined by

$$\nabla^2\chi=\nabla\cdot\mathbf{V} \tag{2}$$

and

$$\nabla^2\psi=\mathbf{k}\cdot\nabla\times\mathbf{V}. \tag{3}$$

These are sufficient to determine $\mathbf{V}$ in an infinite domain. However, in a restricted region the solutions also depend upon external sources whose effects determine the boundary conditions. Solutions may be found with Dirichlet, Neumann, or mixed conditions. Hawkins and Rosenthal [1] have considered various means of satisfying the boundary constraints

$$\mathbf{n}\cdot\mathbf{V}=V_n=-\frac{\partial\psi}{\partial s}+\frac{\partial\chi}{\partial n} \tag{4}$$

and

$$\mathbf{s}\cdot\mathbf{V}=V_s=\frac{\partial\psi}{\partial n}+\frac{\partial\chi}{\partial s}, \tag{5}$$

where $\mathbf{n}$ and $\mathbf{s}$ are normal and tangential unit vectors on the curve C bounding the region A. Sangster [2] has given a variational solution minimizing the kinetic energy due to the potential contribution.

Boundary conditions are determined here such that the areal total velocity, divergence, and vorticity are conserved in the resolution. These conditions would be somewhat simpler in implementation than those currently being used, as well as having more desirable conservation properties. After introducing the analysis method, the divergent and rotational parts of the wind are found separately. They are finally combined in a representation of the total wind field.

2. VARIATIONAL FORMULATION

Suppose that it is desired to modify the observed two dimensional wind $\mathbf{V}^o$ such that the adjusted divergence and vorticity fields satisfy prescribed functions $G(x,y)$ and $F(x,y)$, respectively, while minimizing changes in the observed wind. As suggested by Sasaki [3], this can be done by constructing the difference functional

$$E[\mathbf{V},\nabla\cdot\mathbf{V},\mathbf{k}\cdot\nabla\times\mathbf{V}]=\int_A[(\mathbf{V}-\mathbf{V}^o)^2+2\chi(\nabla\cdot\mathbf{V}-G)+2\psi(\mathbf{k}\cdot\Delta\times\mathbf{V}-F)]dA. \tag{6}$$

Here 2χ and 2ψ are Lagrangian multipliers introducing the constraints. The existence of an actual minimum depends upon the positive definiteness of a matrix associated with the second variation. A necessary condition for its existence is that the Euler-Lagrange equations resulting from the first variation vanish along with the associated natural boundary conditions.

The Euler-Lagrange equations are

$$\mathbf{V}=\mathbf{V}^o+\nabla\chi+\mathbf{k}\times\nabla\psi, \tag{7}$$

$$\nabla\cdot\mathbf{V}=G, \tag{8}$$

and

$$\mathbf{k}\cdot\nabla\times\mathbf{V}=F. \tag{9}$$

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The corresponding natural boundary conditions are

$$\chi\delta V_n|_C=0 \tag{10}$$

and

$$\psi\delta V_s|_C=0. \tag{11}$$

The multiplier χ acts as a potential for the velocity difference ($\mathbf{V}-\mathbf{V}^o$), and ψ is the corresponding stream function.

3. BOUNDARY CONDITIONS FOR THE DIVERGENT WIND

The divergent part of the wind is obtained with the choices $G=\nabla\cdot\mathbf{V}^o$ and $F=0$. The multipliers then satisfy

$$\nabla^2\chi_1=0 \tag{12}$$

and

$$\nabla^2\psi_1=-\mathbf{k}\cdot\nabla\times\mathbf{V}^o \tag{13}$$

subject to (10) and (11). The boundary choice $\chi_1(c)=0$ results in $\chi_1=0$ everywhere. Since

$$\int_A\nabla\cdot(\mathbf{V}-\mathbf{V}^o)dA=\oint_C(V_n-V_n^o)ds=\oint_C\frac{\partial\chi_1}{\partial_n}ds, \tag{14}$$

the total divergence is conserved if the alternate condition $\delta V_n(c)=0$ is satisfied by $V_n=V_n^o$. By analogy with (4), this results in $\partial\chi_1/\partial n=\partial\psi_1/\partial s$, and the divergence is conserved.

The only nontrivial χ-contribution to the irrotational component arises from nonhomogeneous boundary conditions. When the boundary is sufficiently far removed this can be neglected, and the irrotational part of the wind is found as the difference between the observed and solenoidal fields. The argument for the homogenous condition is enhanced by noting that

$$\int_A\nabla\chi_1dA=\oint_C n\chi_1ds. \tag{15}$$

The total velocity is conserved by $\chi_1(c)=0$, and the divergent wind reduces to

$$\mathbf{V}_\chi=\mathbf{V}^o+\mathbf{k}\times\nabla\psi_1. \tag{16}$$

Thus, the divergence is also conserved.

Satisfaction of (11) by $\psi_1(c)=0$ leads to

$$\int_A\mathbf{V}\chi dA-\int_A\mathbf{V}^o dA=\oint_C\psi_1 d\mathbf{s}=0. \tag{17}$$

Only when the observed wind is irrotational would this be acceptable. The natural condition is better satisfied by $\delta V_s(c)=0$. Since

$$V_s-V_s^o=\frac{\partial\psi_1}{\partial n}+\frac{\partial\chi_1}{\partial s}, \tag{18}$$

and the previous choice dictates $\partial\chi_1/\partial s=0$, the selection $V_S=0$ leads to

$$\frac{\partial\psi_1}{\partial n}=-V_s^o. \tag{19}$$

Since

$$\int_A\mathbf{k}\cdot\nabla\times(\mathbf{V}-\mathbf{V}^o)dA=\oint_C(V_s-V_s^o)ds=\oint_C\frac{\partial\psi_1}{\partial n}ds, \tag{20}$$

the total vorticity (circulation) of the adjusted field vanishes.

4. BOUNDARY CONDITIONS FOR THE SOLENOIDAL WIND

The solenoidal part of the wind is determined by the choices $G=0$ and $F=\mathbf{k}\cdot\nabla\times\mathbf{V}^o$. The multipliers for this separate formulation then satisfy

$$\nabla^2\chi_2=-\nabla\cdot\mathbf{V}^o \tag{21}$$

and

$$\nabla^2\psi_2=0. \tag{22}$$

By arguments similar to those above, the homogeneous choice $\psi_2(c)=0$ is made so that the solenoidal wind is

$$\mathbf{V}_\psi=\mathbf{V}^o+\nabla\chi_2. \tag{23}$$

Solution of (21) subject to the natural condition $\chi_2(c)=0$ would yield the correct local vorticity, but it also leads to

$$\int_A\mathbf{V}_\psi dA=\int_A\mathbf{V}^o dA. \tag{24}$$

Instead, (10) is satisfied by taking $V_n=0$. Since $\partial\psi_2/\partial s=0$, this corresponds to

$$\frac{\partial\chi_2}{\partial n}=-\mathbf{V}_n{}^o. \tag{25}$$

The modified field is then nondivergent.

5. CONCLUSIONS

The adjusted wind is given by

$$\mathbf{V}-\mathbf{V}^o=\mathbf{V}^o+\nabla\chi_2+\mathbf{k}\times\nabla\psi_1, \tag{26}$$

where ψ_1 and χ_2 are solutions of (13) and (21), respectively, subject to (25) and (19). With the choices made in satisfying the natural boundary conditions, the components on the boundary are

$$V_n-V_n{}^o=-\frac{\partial\psi_1}{\partial s} \tag{27}$$

and

$$V_s-V_s{}^o=\frac{\partial\chi_2}{\partial s}. \tag{28}$$

These suggest that $\psi_1(c)$ and $\chi_2(c)$ must be constants, say ψ_1^* and χ_2^*, if the adjusted and observed boundary velocities are to agree locally. The choice of constants can be made by noting that

$$\int_A(\mathbf{V}-\mathbf{V}^o)dA=\int_A\mathbf{V}^o dA+\oint_C\mathbf{n}\chi_2ds+\oint_C\psi_1d\mathbf{s}. \tag{29}$$

The total velocity is conserved in the resolution provided that $\psi_1(c)$ and $\chi_2(c)$ satisfy

$$\oint_C \mathbf{n}\chi_2 ds + \oint_C \psi_1 d\mathbf{s} = -\int_A \mathbf{V}^o dA. \tag{30}$$

For constant values $\chi_2{}^*$ and $\psi_1{}^*$ on the boundary of a rectangle of sides L_x and L_y, it is easily shown that

$$\psi_1{}^* = -\frac{A}{2}\frac{L_x\bar{u}^o + L_y\bar{v}^o}{L_x{}^2 + L_y{}^2} \tag{31}$$

$$\chi_2{}^* = -\frac{A}{2}\frac{L_y\bar{u}^o - L_x\bar{v}^o}{L_x{}^2 + L_y{}^2}, \tag{32}$$

where $\bar{u}^o$ and $\bar{v}^o$ are the areal average components of $\mathbf{V}^o$.

A similar consideration of the suggestion by Sangster [2], as implemented by Hawkins and Rosenthal [1], shows that their total solenoidal velocity is taken to equal the observed. This will be in error to the extent that the divergent part of the wind has a non-zero average.

The boundary conditions suggested here are easily obtained and result in useful conservation properties. Since the solution for ψ_1, for example, can be written as

$$\psi_1(x', y') = -\frac{1}{2\pi}\int_A \ln r\,\mathbf{k}\cdot\nabla\times\mathbf{V}^o dA + \frac{1}{2\pi}\oint_C \left[\psi_1 \frac{\partial \ln r}{\partial n} - \ln r \frac{\partial \psi_1}{\partial n}\right] ds,$$

where (x', y') is the point of observation and $r^2 = (x'-x)^2 + (y'-y)^2$, the boundary conditions provide a unique solution.

REFERENCES

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